MA 111 Project #1 – Curve Fitting

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Problem Descriptions

The first problem is to model a ski jump using curve fitting. The ski jump starts off horizontally 80 feet above the bottommost part of the jump. It then slopes downward until it bottoms out at a horizontal distance of 120 feet from the starting position. The jump then heads back up and continues an additional 20 feet horizontally and five feet higher than the bottom of the ski jump.

The second problem is to model a cocktail glass that must be able to accommodate a large cherry or olive of diameter 0.75 inches at the bottom. IT is 3.5 inches tall and three inches across the top. Each lip must make an angle of 30 degrees with the horizontal.

The third problem is to draw a smooth curve through the three points (0, 0), (1, 1), and (2, 0) using cubic splines. The curve must have horizontal tangents at each of the given points.

Analysis of Problem 1

The first step was to define the points under consideration (Subsection 1.2 of Maple sheet). The first point is (0, 80); the second point is (120, 0); the third point is (140, 5). These are obtained from the problem statement. Because the ski jump must begin horizontally, there is a horizontal tangent at (0, 80). Likewise, because the ski jump bottoms out at (120, 0), there is a minimum, and thus a second horizontal tangent there.

These five constraints give us an equation with five coefficients, which renders a fourth-degree polynomial equation in the form of (Subsection 1.1):

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

Because tangents lines must be considered, the derivative must be obtained. It was found to be (Subsection 1.1),

$$y = 4ax^3 + 3bx^2 + 2cx + d$$

Passing through the three points gives the following three equations (Subsection 1.3),

$$eq1 := e = 80$$

$$eq2 := 207360000\,a + 1728000\,b + 14400\,c + 120\,d + e = 0$$

$$eq3 := 384160000\,a + 2744000\,b + 19600\,c + 140\,d + e = 5$$

Setting the tangents at points (0, 80) and (120, 0) to zero gives the following equations,

$$eq4 := d = 0$$

$$eq5 := 6912000\ a + 43200\ b + 240\ c + d = 0$$

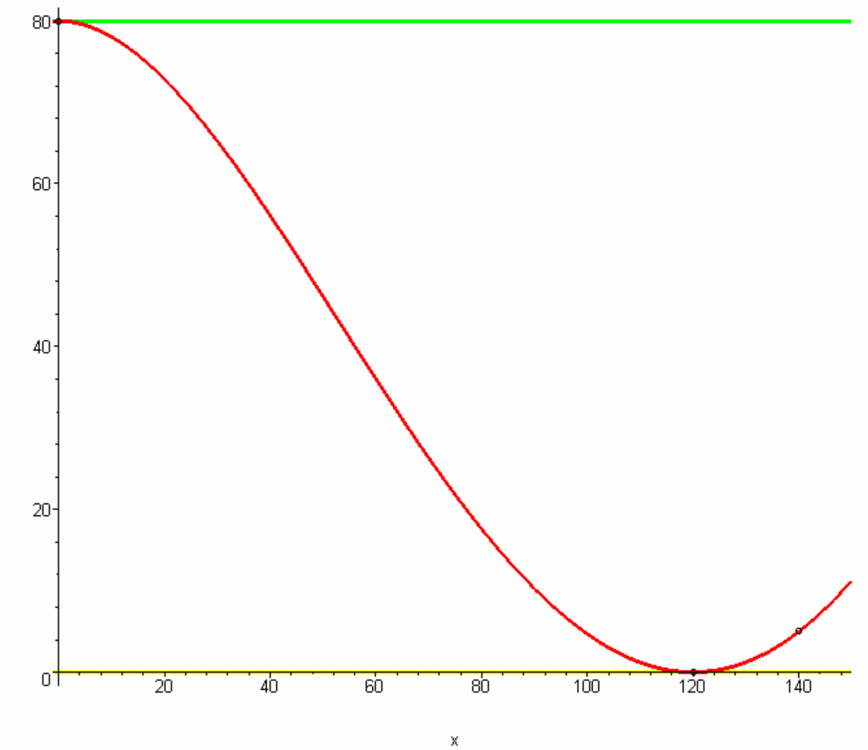
This resolves into a system of five equations containing five unknowns with solution (Subsection 1.4)

$$d = 0,\ a = \frac{-13}{42336000},\ b = \frac{11}{66150},\ c = \frac{-31}{1470},\ e = 80$$

and function

$$ys := -\frac{13}{42336000}x^4 + \frac{11}{66150}x^3 - \frac{31}{1470}x^2 + 80$$

The graph, along with the tangent lines (Subsection 1.5):



Analysis of Problem 2

Again, the first step was to define the points under consideration (Subsection 2.2 of Maple sheet). The first point is (-1.5, 3.5); the second point is (0, 0); the third point is (1.5, 3.5). These are obtained from the problem statement, and the desire to center the cocktail glass around the origin. Because the lip of the cocktail glass must make an angle of 30 degrees with the horizontal, the slope at the endpoint (-1.5, 3.5) must be $-\frac{\sqrt{3}}{3}$, and the slope at the endpoint (1.5, 3.5) must be

$\frac{\sqrt{3}}{3}$. Likewise, there is a horizontal tangent at (0, 0), which is the bottom of the glass.

The additional issue of providing sufficient space for an olive or cherry that does not roll at the bottom of the cocktail glass causes curvature to be necessary to consider. This adds a seventh constraint.

To determine the curvature, which is the value of the second derivative at the point in question, or (0, 0), the second derivative of the equation for a semicircle (the top half) was determined. This equation can be given in the form,

$$\frac{z^2 - 2\,z\,r + r^2}{(x^2 + z^2 - 2\,z\,r + r^2)^{(3/2)}}$$

Taking the second derivative and substituting zero for x and y due to the fact that the point under consideration is (0, 0) gives a constant that is the curvature of both the semicircle and the function to be ultimately obtained. (Subsection 2.6)

These seven constraints give us an equation with seven coefficients, which renders a sixth-degree polynomial equation in the form of (Subsection 2.1):

$$y = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$$

Because tangents lines must be considered, the derivative must be obtained. It was found to be (Subsection 2.1),

$$y = 6ax^5 + 5bx^4 + 4cx^3 + 3dx^2 + 2ex + f$$

In addition, the second derivative must be determined at point (0, 0) to be the curvature value obtained previously. It was found to be (Subsection 2.1),

$$y = 30ax^4 + 20bx^3 + 12cx^2 + 6dx + 2e$$

Passing through the three points gives the following three equations (Subsection 2.3),

$$eq1 := 11.390625\,a - 7.59375\,b + 5.0625\,c - 3.375\,d + 2.25\,e - 1.5\,f + g = 3.5$$

$$eq2 := g = 0$$

$$eq3 := 11.390625\,a + 7.59375\,b + 5.0625\,c + 3.375\,d + 2.25\,e + 1.5\,f + g = 3.5$$

Setting the tangent at point (0, 0) to zero, the tangent at (-1.5, 3.5) to $-\frac{\sqrt{3}}{3}$, and the tangent at (1.5, 3.5) to $\frac{\sqrt{3}}{3}$ gives the following equations,

$$eq4 := -45.56250\,a + 25.3125\,b - 13.500\,c + 6.75\,d - 3.0\,e + f = -\frac{\sqrt{3}}{3}$$

$$eq5 := 45.56250\, a + 25.3125\, b + 13.500\, c + 6.75\, d + 3.0\, e + f = \frac{\sqrt{3}}{3}$$

$$eq6 := f = 0$$

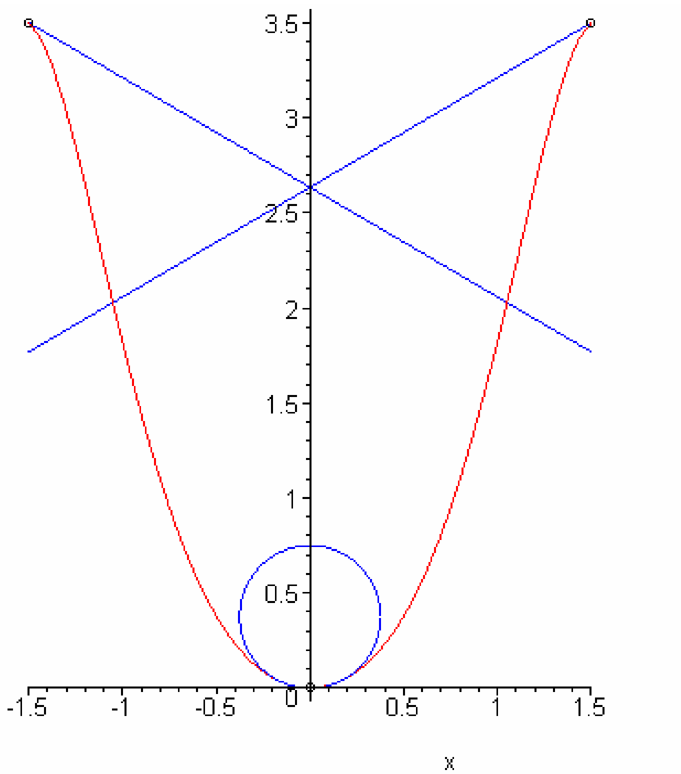
This resolves into a system of seven equations containing seven unknowns with solutions (Subsection 2.4)

$$b = 0., c = 0.8033407009, e = 1.333350000, a = -0.3131478561, g = 0., f = 0., d = 0.$$

and function

$$ys := -0.3131478561\, x^6 + 0.8033407009\, x^4 + 1.333350000\, x^2$$

The graph, along with the tangent lines (Subsection 2.5):



Analysis of Problem 3

The first point under consideration is (0, 0); the second point is (1, 1); the third point is (2, 0). These are obtained from the problem statement. There is a horizontal tangent at each of the given points.

Two cubic equations are required in solving this problem. The first must cover points (0, 0) and (1, 1), while the second must include points (1, 1) and (2, 0). The two points and two tangencies for each function give a total of eight constraints (Subsection 3.1). The form of the cubic equations can be given by

$$y = ax^3 + bx^2 + cx + d$$

$$y = ex^3 + fx^2 + gx + h$$

Because tangents lines must be considered, the derivatives must be obtained. It was found to be (Subsection 3.1),

$$y = 3ax^2 + 2bx + c$$

$$y = 3ex^2 + 2fx + g$$

Passing through the two points gives the following four equations (Subsection 3.3),

$$eq1 := d = 0$$

$$eq2 := a + b + c + d = 1$$

$$eq5 := e + f + g + h = 1$$

$$eq6 := 8\,e + 4\,f + 2\,g + h = 0$$

Setting the tangents at each point to zero gives the following four equations,

$$eq3 := c = 0$$

$$eq4 := 3\,a + 2\,b + c = 0$$

$$eq7 := 3\,e + 2\,f + g = 0$$

$$eq8 := 12\,e + 4\,f + g = 0$$

This resolves into a system of eight equations containing eight unknowns with solutions (Subsection 3.4)

$$a = -2,\ d = 0,\ c = 0,\ b = 3$$

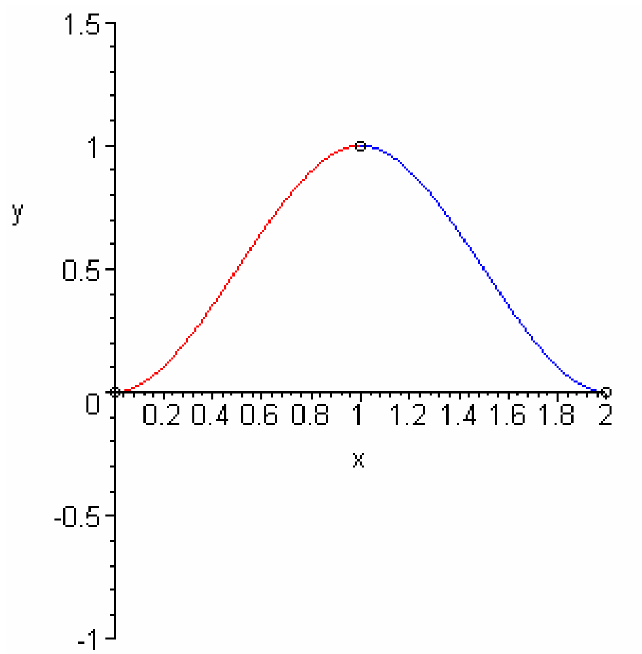
$$g = 12,\ e = 2,\ f = -9,\ h = -4$$

and functions

$$ps := -2\,x^3 + 3\,x^2$$

$$qs := 2\,x^3 - 9\,x^2 + 12\,x - 4$$

The graph, along with the tangent lines (Subsection 3.5):



Conclusions, Problem 1

Using the first derivative to specify horizontal tangents was essential to determining the correct function for the problem. There was no need to use second derivatives, as the concavity or curvature was not an issue in this particular problem.

Conclusions, Problem 2

Given the restraint on the angle of the lip of the cocktail glass, as well as the minimum at the point (0, 0), it was essential to use first derivatives, whether they be set to zero or a trigonometric function. The second derivative likewise became necessary in determining the curvature.

Conclusions, Problem 3

The required use of cubic splines essentially doubled the number of unknowns for which to solve. Using cubic splines meant that the first derivatives and values at the meeting point must be the same; in the case of the first derivatives, this was zero, and in the case of the values, it was one (1), because the point at which they joined was (1, 1). Second derivatives were unnecessary, as curvature or concavity were not relevant.

1. Ski Jump

1.1 Definition of functions and derivatives

```
> restart;
  # set up curve and points
  y := a*x^4+b*x^3+c*x^2+d*x + e;
  dy := diff(y,x);
```

$$y := a\,x^4 + b\,x^3 + c\,x^2 + d\,x + e$$

$$dy := 4\,a\,x^3 + 3\,b\,x^2 + 2\,c\,x + d$$

1.2 Definition of points and tangencies

```
> P1 := [0,80];
  P2 := [120,0];
  P3 := [140,5];
  T1 := [0, 0];
  T2 := [120, 0];
```

$$P1 := [0, 80]$$

$$P2 := [120, 0]$$

$$P3 := [140, 5]$$

$$T1 := [0, 0]$$

$$T2 := [120, 0]$$

1.3 Substitution to obtain coefficients

```
> eq1 :=  subs(x=P1[1],y) = P1[2];
  eq2 :=  subs(x=P2[1],y) = P2[2];
  eq3 :=  subs(x=P3[1],y) = P3[2];
  eq4 :=  subs(x=T1[1],dy) = T1[2];
  eq5 :=  subs(x=T2[1],dy) = T2[2];
```

$$eq1 := e = 80$$

$$eq2 := 207360000\,a + 1728000\,b + 14400\,c + 120\,d + e = 0$$

$$eq3 := 384160000\,a + 2744000\,b + 19600\,c + 140\,d + e = 5$$

$$eq4 := d = 0$$

$$eq5 := 6912000\,a + 43200\,b + 240\,c + d = 0$$

1.4 Solve equations to obtain function

```
> csol := solve({eq1,eq2,eq3, eq4, eq5},{a,b,c,d,e});
> ys := subs(csol,y);
  t1 := P1[2] + T1[2]*(x-P1[1]);
  t2 := P2[2] + T2[2]*(x-P3[1]);
```

$$csol := \{d = 0,\ a = \frac{-13}{42336000},\ b = \frac{11}{66150},\ c = \frac{-31}{1470},\ e = 80\}$$

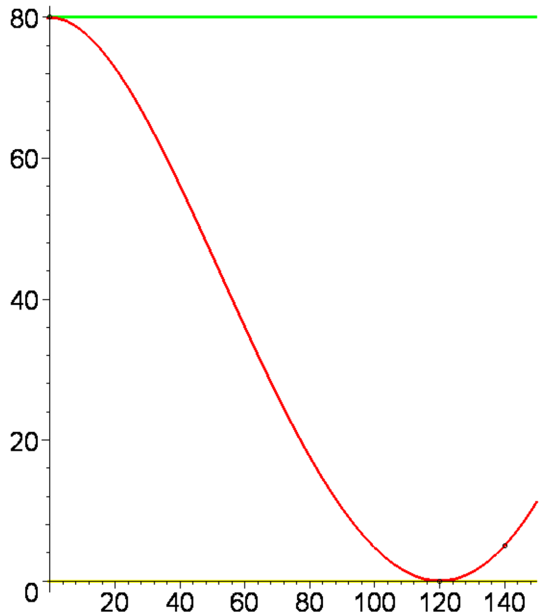
$$ys := -\frac{13}{42336000}\,x^4 + \frac{11}{66150}\,x^3 - \frac{31}{1470}\,x^2 + 80$$

$$t1 := 80$$

$$t2 := 0$$

1.5 Graph

```
> with(plots):
  display({plot([ys, t1, t2],x= -1..150, thickness=3),plot([P1,P2,P3], color=black, style=point, symbol=circle)});
Warning, the name changecoords has been redefined
```



2. Cocktail Glass

2.1 Definition of functions and constants

```
> restart;
  # set up curve and points
  y := a*x^6+b*x^5+c*x^4+d*x^3+e*x^2+f*x+g;
  dy := diff(y,x);
  d2y := diff(y,x,x);
  r := 0.375;
```

$$y := a\,x^6 + b\,x^5 + c\,x^4 + d\,x^3 + e\,x^2 + f\,x + g$$
$$dy := 6\,a\,x^5 + 5\,b\,x^4 + 4\,c\,x^3 + 3\,d\,x^2 + 2\,e\,x + f$$
$$d2y := 30\,a\,x^4 + 20\,b\,x^3 + 12\,c\,x^2 + 6\,d\,x + 2\,e$$
$$r := 0.375$$

2.2 Definition of points and curvature

```
> P1 := [-1.5,3.5];
  P2 := [0, 0];
  P3 := [1.5,3.5];
  T1 := [-1.5, tan(-Pi/6)];
  T2 := [1.5, tan(Pi/6)];
  T3 := [0, 0];
  C1 := [0,2.6667];
```

$$P1 := [-1.5, 3.5]$$
$$P2 := [0, 0]$$
$$P3 := [1.5, 3.5]$$
$$T1 := \left[-1.5, -\frac{\sqrt{3}}{3}\right]$$
$$T2 := \left[1.5, \frac{\sqrt{3}}{3}\right]$$
$$T3 := [0, 0]$$
$$C1 := [0, 2.6667]$$

2.3 Substitution for coefficients

```
> eq1 :=  subs(x=P1[1],y) = P1[2];
  eq2 :=  subs(x=P2[1],y) = P2[2];
  eq3 :=  subs(x=P3[1],y) = P3[2];
  eq4 :=  subs(x=T1[1],dy) = T1[2];
  eq5 :=  subs(x=T2[1],dy) = T2[2];
  eq6 :=  subs(x=T3[1],dy) = T3[2];
  eq7 :=  subs(x=C1[1],d2y) = C1[2];
```

$$eq1 := 11.390625\,a - 7.59375\,b + 5.0625\,c - 3.375\,d + 2.25\,e - 1.5\,f + g = 3.5$$
$$eq2 := g = 0$$
$$eq3 := 11.390625\,a + 7.59375\,b + 5.0625\,c + 3.375\,d + 2.25\,e + 1.5\,f + g = 3.5$$
$$eq4 := -45.56250\,a + 25.3125\,b - 13.500\,c + 6.75\,d - 3.0\,e + f = -\frac{\sqrt{3}}{3}$$

$$eq5 := 45.56250\,a + 25.3125\,b + 13.500\,c + 6.75\,d + 3.0\,e + f = \frac{\sqrt{3}}{3}$$

$$eq6 := f = 0$$

$$eq7 := 2\,e = 2.6667$$

2.4 Solving for equation of curve and olive/cherry

```
> csol := solve({eq1,eq2,eq3,eq4,eq5,eq6,eq7},{a,b,c,d,e,f,g});
> ys := subs(csol,y);
  r := 0.375;
  bcirc := r-sqrt(r^2-x^2);
  tcirc := r+sqrt(r^2-x^2);
  t1 := P1[2] + T1[2]*(x-P1[1]);
  t2 := P3[2] + T2[2]*(x-P3[1]);
```

$$csol := \{ b = 0.,\, c = 0.8033407009,\, e = 1.333350000,\, a = -0.3131478561,\, g = 0.,\, f = 0.,\, d = 0. \}$$

$$ys := -0.3131478561\,x^6 + 0.8033407009\,x^4 + 1.333350000\,x^2$$

$$r := 0.375$$

$$bcirc := 0.375 - \sqrt{0.140625 - x^2}$$

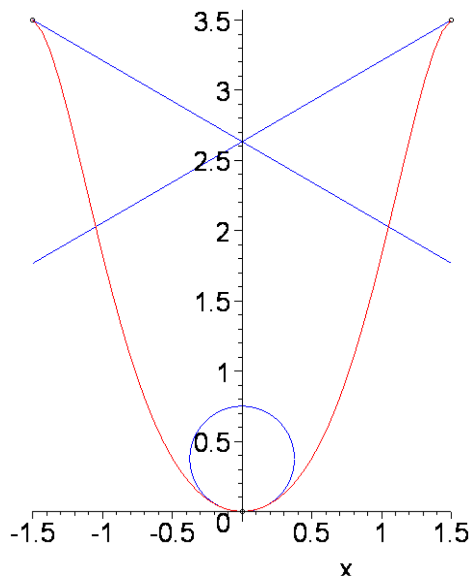
$$tcirc := 0.375 + \sqrt{0.140625 - x^2}$$

$$t1 := 3.5 - \frac{\sqrt{3}\,(x + 1.5)}{3}$$

$$t2 := 3.5 + \frac{\sqrt{3}\,(x - 1.5)}{3}$$

2.5 Graph

```
> with(plots):
  display({plot([ys,t1,t2],x= -1.5..1.5, color=[red,blue,blue]),
           plot([tcirc,bcirc], x= -r..r, color= blue),
           plot([P1,P2,P3], color=black, style=point, symbol=circle)},
           scaling = constrained );
Warning, the name changecoords has been redefined
```



2.6 Determination of Curvature

```
> semicircle := sqrt((x)^2 + (z - r)^2);
  simplify(diff(semicircle, x, x));
```

$$semicircle := \sqrt{x^2 + (z - 0.375)^2}$$

$$\frac{8.\,(64.\,z^2 - 48.\,z + 9.)}{(64.\,x^2 + 64.\,z^2 - 48.\,z + 9.)^{(3/2)}}$$

```
>
```

3. Cubic Splines

3.1 Definition of functions

```
> restart;
  # set up curve and points
  p := a*x^3+b*x^2+c*x+d;
  dp := diff(p,x);
```

$$p := a\,x^3 + b\,x^2 + c\,x + d$$

$dp := 3\,a\,x^2 + 2\,b\,x + c$

```
> q := e*x^3+f*x^2+g*x+h;
  dq := diff(q,x);
```

$q := e\,x^3 + f\,x^2 + g\,x + h$

$dq := 3\,e\,x^2 + 2\,f\,x + g$

3.2 Definition of points and tangencies

```
> P1p := [0,0];
  P2p := [1,1];
  T1p := [0, 0];
  T2p := [1, 0];
```

$P1p := [0, 0]$

$P2p := [1, 1]$

$T1p := [0, 0]$

$T2p := [1, 0]$

```
> P1q := [1,1];
  P2q := [2,0];
  T1q := [1,0];
  T2q := [2,0];
```

$P1q := [1, 1]$

$P2q := [2, 0]$

$T1q := [1, 0]$

$T2q := [2, 0]$

3.3 Substitution to obtain coefficients

```
> eq1 :=  subs(x=P1p[1],p) = P1p[2];
  eq2 :=  subs(x=P2p[1],p) = P2p[2];
  eq3 :=  subs(x=T1p[1],dp) = T1p[2];
  eq4 :=  subs(x=T2p[1],dp) = T2p[2];
```

$eq1 := d = 0$

$eq2 := a + b + c + d = 1$

$eq3 := c = 0$

$eq4 := 3\,a + 2\,b + c = 0$

```
> eq5 :=  subs(x=P1q[1],q) = P1q[2];
  eq6 :=  subs(x=P2q[1],q) = P2q[2];
  eq7 :=  subs(x=T1q[1],dq) = T1q[2];
  eq8 :=  subs(x=T2q[1],dq) = T2q[2];
```

$eq5 := e + f + g + h = 1$

$eq6 := 8\,e + 4\,f + 2\,g + h = 0$

$eq7 := 3\,e + 2\,f + g = 0$

$eq8 := 12\,e + 4\,f + g = 0$

3.4 Solve for equations

```
> csolp := solve({eq1,eq2,eq3,eq4},{a,b,c,d});
> ps := subs(csolp,p);
```

$csolp := \{a = -2, d = 0, c = 0, b = 3\}$

$ps := -2\,x^3 + 3\,x^2$

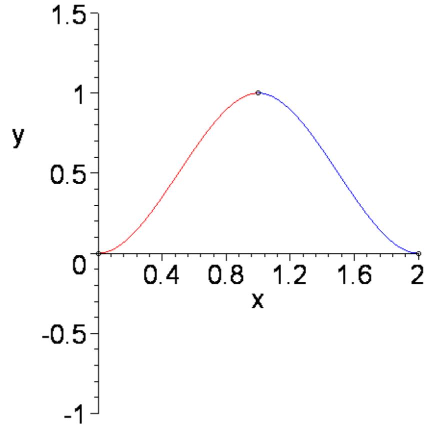
```
> csolq := solve({eq5,eq6,eq7,eq8},{e,f,g,h});
  qs := subs(csolq,q);
```

$csolq := \{g = 12, e = 2, f = -9, h = -4\}$

$qs := 2\,x^3 - 9\,x^2 + 12\,x - 4$

3.5 Graph

```
> with(plots):
  display({plot(ps,x= 0..1, y=-1..1.5),
           plot(qs,x= 1..2, y=-1..1.5, color=blue),
           plot([P1p,P2p,P2q], color=black, style=point, symbol=circle)},
           scaling = constrained );
Warning, the name changecoords has been redefined
```


1.5
1
0.5
0
-0.5
-1
y
0.4
0.8
1.2
1.6
2
x

>

>